Problems on Algebra

COMPLEX NUMBERS

1. Show that z_1, z_2 are the inverse points with respect to the line

$$z\bar{a}+a\bar{z} = b \text{ if } z_1\bar{a}+a\bar{z}_2 = b$$

2. Show that the triangle whose vertices are z_1, z_2, z_3 and z_1', z_2', z_3' ar equilateral, if

$$(z_1-z_2)(z_1'-z_2') = (z_2-z_3)(z_2'-z_3') = (z_3-z_1)(z_3'-z_1')$$

3. Show that the point a' is the reflection of the point a in the line

$$z\bar{b}+\bar{z}b+c = 0, \text{ if } a'\bar{b}+\bar{a}b+c = 0$$

4. Prove that $\left|\dfrac{z-1}{z+1}\right| =$ constant and amp $\left(\dfrac{z-1}{z+1}\right) =$ constant ar orthogonal circles.

5. If $z_1 = 2+5i$, $z_2 = 3-i$ find (a) $z_1\,o\,z_2$, (b) $z_1\times z_2$, (c) $z_2\,o\,z_1$, (d) $z_2\times z_1$ (acute angle between z_1 and z_2.

6. Prove that

$$\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\cos\frac{6\pi}{2n+1}+\dots+\cos\frac{2n\pi}{2n+1} = -\frac{1}{2}$$

7. Write equations whose roots are equal to numbers

(a) $\sin^2\dfrac{\pi}{2n+1}, \sin^2\dfrac{2\pi}{2n+1}, \sin^2\dfrac{3\pi}{2n+1}, \dots\dots, \sin^2\dfrac{n\pi}{2n+1}$

(b) $\cot^2\dfrac{\pi}{2n+1}, \cot^2\dfrac{2\pi}{2n+1}, \cot^2\dfrac{3\pi}{2n+1}, \dots., \cot^2\dfrac{n\pi}{2n+1}$

8. Simplify the expressions of the sums

(a) $\cot^2\dfrac{\pi}{2n+1}+\cot^2\dfrac{2\pi}{2n+1}+\cot^2\dfrac{3\pi}{2n+1}+\dots+\cot^2\dfrac{n\pi}{2n+1}$

(b) $\csc^2\dfrac{\pi}{2n+1}+\csc^2\dfrac{2\pi}{2n+1}+\csc^2\dfrac{3\pi}{2n+1}+\dots+\csc^2\dfrac{n\pi}{2n+1}$

9. Simplify the expressions of the products

(a) $\sin\dfrac{\pi}{2n+1}\cdot\sin\dfrac{2\pi}{2n+1}\cdot\sin\dfrac{3\pi}{2n+1}\cdot\dots\dots\sin\dfrac{n\pi}{2n+1}$

and $\sin\dfrac{\pi}{2n}\sin\dfrac{2\pi}{2n}\sin\dfrac{3\pi}{2n}\dots\sin\dfrac{(n-1)\pi}{2n}$

(b) $\cos\dfrac{\pi}{2n+1}\cdot\cos\dfrac{2\pi}{2n+1}\cdot\cos\dfrac{3\pi}{2n+1}\cdot\dots\dots\cos\dfrac{n\pi}{2n+1}$

and $\cos\dfrac{\pi}{2n}\cos\dfrac{2\pi}{2n}\cos\dfrac{3\pi}{2n}\dots.\cos\dfrac{n\pi}{2n}.$

10. Prove that if m, n and p are arbitrary integers then the expression

$$\sin\frac{m\pi}{p}\sin\frac{n\pi}{p}+\sin\frac{2m\pi}{p}\sin\frac{2n\pi}{p}+\sin\frac{3m\pi}{p}\sin\frac{3n\pi}{p}+\ldots\ldots$$
$$+\ldots\ldots+\sin\frac{(p-1)\,m\pi}{p}\sin\frac{(p-1)\,n\pi}{p}$$

is equal to $-p/2$ when $(m+n)$ is divisible by $2p$ and $m-n$ is not, is equal to $p/2$ when $m-n$ is divisible by $2p$ and $(m+n)$ is not and equal to zero when both $(m+n)$ and $(m-n)$ are divisible by $2p$ or not divisible by $2p$.

11. Show that from the results established in problems 8 (a) and 8 (b) it follows that for any positive integer n the value of the sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots.+\frac{1}{n^2}\quad\text{lies between}$$

$$\left(1-\frac{1}{2n+1}\right)\left(1-\frac{2}{2n+1}\right)\frac{\pi^2}{6}\ \text{and}\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{2n+1}\right)\frac{\pi^2}{6}$$

12. Find the sum of the 50th powers of all the sides and of all the diagonals of a regular 100–gon inscribed in a circle of radius R.

13. It is known that $\left|z+\frac{1}{z}\right|=a$, where z is a complex number. What are the greatest and least positive values of the modulus $|z|$ of the complex number z ?

14. Let a sum of n complex numbers be equal to zero. Prove that among them there are two numbers whose arguments differ by not less than 120°.

Is it possible to replace in this problem the angle of 120° by a smaller angle ?

15. Let $c_1, c_2, \ldots\ldots, c_n$ and z be complex numbers such that

$$\frac{1}{z-c_1}+\frac{1}{z-c_2}+\ldots\ldots+\frac{1}{z-c_n}=0$$

Prove that if the numbers $c_1, c_2, \ldots, c_n$ are represented in the complex plane by the vertices of a convex n–gon then the number z is represented by a point lying inside that n–gon.

16. Let $\alpha=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$ (n, a positive integer)

and let $A_k=x+y\,\alpha^k+z\,\alpha^{2k}+\ldots+w\,\alpha^{(n-1)k}$

$(k=0, 1, 2, \ldots, n-1)$

where $x, y, z, \ldots., u, w$ are n arbitrary complex numbers. Prove that

$$\sum_{k=0}^{n-1}|A_k|^2=n\{|x|^2+|y|^2+|z|^2+\ldots.\ |w|^2\}$$

17. Prove the following identities

(i) $$\sqrt[3]{\left(\cos\frac{2\pi}{7}\right)}+\sqrt[3]{\left(\cos\frac{4\pi}{7}\right)}+\sqrt[3]{\left(\cos\frac{8\pi}{7}\right)}=\sqrt[3]{\frac{1}{2}(5-3\cdot\sqrt[3]{7})}$$

(ii) $\sqrt[3]{\left(\cos\frac{2\pi}{9}\right)} + \sqrt[3]{\left(\cos\frac{4\pi}{9}\right)} + \sqrt[3]{\left(\cos\frac{8\pi}{9}\right)}$

$$= \sqrt[3]{\frac{1}{2}(3\cdot\sqrt[3]{9}-6)}$$

18. If $(a+\omega)^{-1}+(b+\omega)^{-1}+(c+\omega)^{-1}+(d+\omega)^{-1}=2\omega^{-1}$, $(a+\omega')^{-1}+(b+\omega')^{-1}+(c+\omega')^{-1}+(d+\omega')^{-1}=2(\omega')^{-1}$ where ω and ω' are the imaginary cube roots of unity, prove that

$$(a+1)^{-1}+(b+1)^{-1}+(c+1)^{-1}+(d+1)^{-1} = 2.$$

19. If $\omega = \frac{1}{2}(-1+i\sqrt{3})$. being an imaginary cube root of unity, and a, b, c are real then

$$\sqrt{(a+\omega b+\omega^2 c)} = \pm\frac{1}{2}\{\sqrt{(2D+L)}+i\sqrt{(2D-L)}\}$$

$$\sqrt{(a+\omega^2 b+\omega c)} = \pm\frac{1}{2}\{\sqrt{(2D+L)} - i\sqrt{2D-L}\}$$

where

$$D = \sqrt{(a^2+b^2+c^2-ab-bc-ca)}$$

$$L = 2a-b-c$$

provided that $b > c$. If $b < c$, the sign of i must be changed.

20. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n, nth roots of unity,

(a) Prove that

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1}) = n$$

(b) Hence prove that

$$\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot\sin\frac{3\pi}{n}\cdot\ldots\cdot\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

21. If $|a_n| < 2$ for $1 \le n \le N$, then show that these doesn't exist any z in the circle $|z| < \frac{1}{3}$ such that $1+a_1 z+a_2 z^2+\ldots+a_N z^N = 0$.

22. Show that the two points z_1, z_2 are the inverse points with respect to the circle

$$z\bar{z}+a\bar{z}+\bar{a}z+b = 0$$

if

$$z_1\bar{z}_2+a\bar{z}_2+\bar{a}z_1+b = 0$$

23. Find all complex numbers z for which $|z| = 2\sqrt{2}$ and $\arg\left(\frac{z-(5+i)}{z-(1+i)}\right) = \frac{\pi}{4}$

24. (i) If $z_1, z_2, \ldots, z_7$ are the roots of $(z+1)^7+z^7 = 0$ find the value of $\sum_{k=1}^{7}\mathrm{Re}(z_k)$

(ii) If m and x are real numbers, find the value of

$$e^{2mi\cot^{-1}x}\cdot\left(\frac{xi+1}{xi-1}\right)^m,$$

25. If ω is the nth root of unity and z_1 and z_2 any two complex numbers. Prove that

$$\sum_{k=0}^{n-1}|z_1+\omega^k z_2|^2 = n\{|z_1|^2+|z_2|^2\} \quad (n \in N)$$

26. Show that inverse of a point a, with respect to the circle $|z-c| = R$ (circle of radius R with centre at the point c) is the point

$$c+\frac{R^2}{\bar{a}-\bar{c}}$$

27. Let $\sum_{i=1}^{4} a_i = 0$ and $\sum_{j=1}^{4} a_j z_j = 0$ prove that z_1, z_2, z_3, z_4 are concyclic if

$$a_1 a_2 |z_1 - z_2|^2 = a_3 a_4 |z_3 - z_4|^2$$

28. (a) Show that the perpendicular distance of a point d (d is a complex number) on the Argand plane from the line $a\bar{z}+\bar{a}z+b = 0$ (a is complex and b is real) is

$$\frac{|a\bar{d}+\bar{a}d+b|}{2|a|}.$$

(b) Find the equation of the circle in complex form which touches the line $iz+\bar{z}+1+i=0$ and the lines $(2-i)z=(2+i)\bar{z}$ and $(2+i)z+(i-2)\bar{z}-4i=0$ are the normals of the circle.

29. If z is a complex number and $\theta_0, \theta_1, \theta_2, \dots, \theta_{2n}$ are real then prove that all the roots of the equation

$$\sin\theta_0 z^{2n}+\cos\theta_1 z^{2n-1}+\sin\theta_2 z^{2n-2}+\dots+\cos\theta_{2n-1}z+\sin\theta_{2n} = b$$

lies in the region

$$|z| > 1-\frac{1}{|b|}$$

30. If $|z| \le 1$ and $|w| \le 1$, show that

$$|z-w|^2 \le (|z|-|w|)^2+(\arg z-\arg w)^2$$

31. A triangle with vertices represented by z_1, z_2 and z_3 has the opposite sides by lengths in the ratio $2:\sqrt{6}:\sqrt{3}-1$ respectively prove that

$$(z_3-z_2)^2+(\sqrt{3}+1)(z_1-z_2)(z_3-z_2)+(\sqrt{3}+1)^2(z_1-z_2)^2 = 0$$

32. Solve $z^7 = 1$, use this to obtain an equation whose roots are

$$\cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \cos\frac{6\pi}{7}$$

Hence conclude that

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = -\frac{1}{2}$$

33. Evaluate

$$\sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right\}^{p}$$

34. Prove that if z_1, z_2 are two complex numbers and $c > 0$. then $|z_1+z_2|^2 \le (1+c)|z_1|^2+\left(1+\frac{1}{c}\right)|z_2|^2$

35. Find the integral solutions of the equation $(1-i)^x = 2^x$

36. Solve the following system of equations in terms of complex numbers

$$z^{13}w^{19} = 1$$
$$z^{5}w^{7} = 1$$
$$z^{2}+w^{2} = -2$$

37. Two different non parallel lines meet the circle $|z| = r$ in the points a, b and c, d respectively. Prove that these lines meet in the point z given by

$$z = \frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

38. Solve the equation $z^7+1 = 0$ and deduce that

(i) $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$

(ii) $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$

(iii) $\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right) = \frac{1}{8}$

(iv) $\tan\frac{\pi}{14}\tan\frac{3\pi}{14}\tan\frac{5\pi}{14} = \frac{1}{\sqrt{7}}$

Also show that

$$(1+y)^7-(1-y)^7 = 14\left(y^2+\tan^2\frac{\pi}{14}\right)\left(y^2+\tan^2\frac{3\pi}{14}\right)\left(y^2+\tan^2\frac{5\pi}{14}\right)$$

and then deduce that

$$\tan^2\left(\frac{\pi}{14}\right)+\tan^2\left(\frac{3\pi}{14}\right)+\tan^2\left(\frac{5\pi}{14}\right) = 5.$$

39. If z_1, z_2 and z_3 are three complex numbers, then prove that

$$z_1 I_m(\bar{z}_2 z_3)+z_2 I_m(\bar{z}_3 z_1)+z_3 I_m(\bar{z}_1 z_2) = 0$$

40. Let z_1, z_2 be any two complex numbers and a, b be two real numbers such that

$a^2+b^2 \neq 0$, Prove that

$$|z_1|^2+|z_2|^2-|z_1^2+z_2^2| \le \frac{2|az_1+bz_2|^2}{(a^2+b^2)} \le |z_1|^2+|z_2|^2 + |z_1^2+z_2^2|$$